Mansoura University
Faculty of Engineering
Math. & Eng. Physics Dept.

Advanced Numerical Analysis

September 8, 2013

2nd year Postgraduate Students

Time allowed: 3 Hours

Prof. Dr. Magdi S. El-Azab
Dr. Waleed Raslan

Answer the following problems (Full mark 100 pts)

1. **(a)** [5 pts] Give the definition of the fractional derivative operator (of Caputo sense) $D^{\alpha}f(t)$, $t>0$, $\alpha\in(0,1)$, discuss its existence and then discuss the validity of the following relations

- $D^{\alpha}D^{\beta}f(t)=D^{\alpha+\beta}f(t),\quad \alpha+\beta\in(1,2)$
- $D^{\alpha}I^{\beta}f(t)=I^{\beta-\alpha}f(t),\quad \beta>\alpha$
- $D^{\alpha}I^{\alpha}f(t)=I^{\alpha}D^{\alpha}f(t)=f(t)$

(b) [5 pts] Give the definition of the Riemann-Liouville fractional derivative operator ${}_{*}D^{\alpha}f(t)$, $t>0$, discus its existence and prove that for any constant $k\neq 0$, then ${}_{*}D^{\alpha}k\neq 0$, while $D^{\alpha}k=0$.

(c) [5 pts] Using two methods, find the solution of the fractional initial value problem $D^{\alpha}x(t)=\mu x(t)$, $t>0$, $\alpha\in(0,1)$, $x(0)=x_0$, then prove that this solution satisfies the problem. Also, study the continuation of the solution ($\alpha\to 1$).

---

2. Given the problem $-\Delta u+5u=f$ in Ω, $u=0$ on $\partial\Omega$, where f is a smooth function.

(a) [5 pts] Define the space of solution.

(b) [5 pts] Write the weak formula corresponding to the given problem.

(c) [5 pts] State and prove Lax-Milgram theorem for this problem.

(d) [5 pts] Write the matrix form of the solution of the problem by Ritz method with the basis $\varphi_1, \varphi_2, \varphi_3, \cdots, \varphi_n$.

---

3. [5 pts] State without proof:

(i) The discrete and continuous forms of Gronwall's inequality,

(ii) Schwarz and Young's inequalities.

---

4. Given the problem $u_t - 4u_{xx} = f$ in $Q_T \equiv (0,\ 1) \times (0,\ T)$, where f is a smooth enough function and $Q_T = \Omega \times I$, $\Omega \equiv (0,\ 1)$, $\partial\Omega \equiv \{0,\ 1\}$, $I \equiv (0, T)$. The problem is associated with the Dirichlit boundary condition

$$u = 0, \qquad (x, t) \in \partial\Omega \times I,$$

and the initial condition $u(x, 0) = \sin \pi x$, $\quad x \in \Omega$.

(a) [5 pts] What is the weak solution of this problem?

(b) [5 pts] Define Roth's solution and the corresponding step functions.

(c) [5 pts] Derive the a priori estimates of the approximate solution.

(d) [5 pts] Discuss the convergence and the error estimates of the approximate solution.

---

5. [10 pts] Apply Galerkin method to solve the boundary value problem

$$-u'' = \cos x, \quad u(0) = 0, \quad u(1) = 0,$$

with the basis

$$\sin x,\ \frac{\sin 2x}{2},\ \frac{\sin 3x}{3},\ \frac{\sin 4x}{4},\ \cdots$$

---

6. [10 pts] For all real numbers a_i, $i = 0, 1, \cdots, n$, prove that

$$2\sum_{i=1}^{n} a_i(a_i - a_{i-1}) = a_n^2 - a_0^2 + \sum_{i=1}^{n}(a_i - a_{i-1})^2$$

---

7. [20 pts] Let Ω be a rectangle $ABCD$ given in the following figure. Consider the problem $-\Delta u + 4u = f(x, y)$ in Ω, associated with the boundary conditions

$$u = 0, \quad \text{on } AD, \qquad u = 5 \quad \text{on } BC,$$

$$\frac{\partial u}{\partial y} + u = 5, \quad \text{on } DC, \qquad \frac{\partial u}{\partial y} + 2u = 6 \quad \text{on } AB$$

Write the weak formula for this boundary value problem. Discuss the existence and uniqueness of the approximate solution.