ANALYSIS OF TRANSIENT HEAT CONDUCTION IN A 2-D RECTANGULAR FIN WITH CONSTANT HEAT FLUX

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التوصيل الحراري الغير مستقر ثنائي البعد
في زعنفه مستطيله تتعرض لفيض حراري ثابت

يهدف البحث الى الحصول على التطور الزمني لتوزيع درجات الحراره في اتجاهين متعامدين نتيجة لانتقال الحراره بالتوصيل الغير مستقر في زعنفه مستطيلة المقطع تتعرض قاعدتها لفيض حراري ثابت ، كما افترض ان الطرف الحر للزعنفه يبرد بالحمل ولكن بمعدل يختلف عن معدل التبريد بالحمل من السطحيين العوديين على سطح القاعده . ونظرا لصعوبة الحصول على حل رياضي تحليلي لهذه المشكله فلقد تم تقسيمها الى جزئين ، في الجزء الاول تم الحصول على التطور الزمني لدرجات الحراره بطريقة الفروق المحدوده في صورة معادلات جبريه بدون ابعاد ، وفي الجزء الثاني تم الحصول على حل رياضي تحليلي كامل لهذه المشكله في حالة الاستقرار حتى يمكن مقارنة نتائج الحل العددي في الازمنه الطويله مع النتائج التي نحصل عليها من الحل التحليلي للحاله المستقره .
وتدل النتائج على ان عدد Biot يؤثر في توزيع درجات الحراره وتطورها الزمني تاثيرا حاسما ،وقد وجد ان نسبة سمك الزعنفه الى عرضها يؤثر كثيرا على توزيع درجات الحراره حيث ترتفع درجات الحراره مع زيادة هذه النسبه ويزداد زمن الوصول الى حالة الاستقرار، كما تدل النتائج على ان الفرق بين توزيع درجات الحراره على سطح الزعنفه و محورها يزداد بزيادة عدد Biot وزيادة نسبه سمك الزعنفه الى عرضها . و لقد تم الحصول عليها تغير توزيع درجات الحراره على سطح الامتصاص لسخان شمسي مع عدد Biot حيث تصل نسبة السمك الى الطول ١٠ .

ABSTRACT

The objective of this work is to investigate the 2-D heat conduction in a convectively cooled rectangular fin whose base is subjected to constant heat flux with different fin tip conditions. For this purpose an analytical steady-state solution is obtained based on the traditional approach (separation of variables). Solution of the transient case of this problem is obtained by applying a new developed dimensionless finite difference technique.

INTRODUCTION

Many fin problems have been studied for one-dimensional case with simple boundary and initial conditions. Analytical steady-state solutions for some cases with simple boundary conditions are reported in [1,2]. Recently, MA et al. [3] obtained solution for the 2-D steady-state problem in a rectangular fin which is convectively cooled with variable heat transfer coefficient. The heat conduction problem in semi-infinite solid with temperature-dependent material properties has been studied in [4]. Transient heat conduction for some simple geometries and initial and boundary conditions has been investigated in [1-2]. Transient conduction in a 2-D fin, which is convectively cooled and its base is subjected to a sudden step temperature change has been investigated in [5,6].

The problem of heat conduction in a 2-D rectangular fin, which is convectively cooled and its base is subjected suddenly to a constant heat flux has not been investigated before. This case is widely applied in the field of solar energy since the absorber surface serves as a fin on the absorber tubes [7]. In this study, a regular separation of variable technique is applied to the steady-state case of this problem. The transient case of this problem is attacked by applying a modified dimensionless finite difference technique which is similar to that developed by the author and repotred in [6].

PROBLEM FORMULATION AND SOLUTION

Consider the 2-D rectangular fin (2b'xL) where both the upper and lower surfaces dissipate heat by to a surrounding at a temperature T^*_∞ with constant heat transfer coefficient h, as shown in Fig.1a. Furthermore, the tip of the fin has a constant heat transfer coefficient h_1 and the base of the fin is subjected to a constant heat flux q_o. For constant thermophysical material properties and no heat generation, the steady-state governing equation for the two-dimensional fin is:

$$\frac{\partial^2 T^*}{\partial x^{*2}} + \frac{\partial^2 T^*}{\partial y^{*2}} = 0 \tag{1}$$

and the boundary conditions are:

$$\left.\frac{\partial T^*}{\partial y^*}\right]_{y^*=0} = 0, \qquad -k \left.\frac{\partial T^*}{\partial y^*}\right]_{y^*=b'} = h\left(T^*\big|_{y^*=b'} - T^*_\infty\right)$$

$$-k \left.\frac{\partial T^*}{\partial x^*}\right]_{x^*=0} = q_o, \qquad -k \left.\frac{\partial T^*}{\partial x^*}\right]_{x^*=L} = h_1\left(T^*\big|_{x^*=L} - T^*_\infty\right)$$

Defining the following dimensionless quantities,

$$x = x^*/L, \quad y = y^*/b, \quad \varepsilon' = b'/L, \quad H' = hb'/k, \quad H_1 = h_1L/k,$$

$$T = \frac{T^* - T^*_\infty}{(q_oL/K)}, \qquad Bi = hL/k,$$

equation (1) can be transformed into the following dimensionless form:

$$\frac{\partial^2 T}{\partial x^2} + \frac{1}{\varepsilon'^2}\frac{\partial^2 T}{\partial y^2} = 0 \tag{3}$$

The boundary conditions are:

$$\frac{\partial T(x,0)}{\partial y} = 0 \tag{4a}$$

$$\frac{\partial T(x,1)}{\partial y} = -H'\,T(x,1) = \varepsilon'\,(Bi/2)\,T(x,1) \tag{4b}$$

$$\frac{\partial T(0,y)}{\partial x} = -1 \tag{4c}$$

$$\frac{\partial T(1,y)}{\partial x} = -H_1\, T(1,y) \tag{4d}$$

Let $$T(x,y) = Y(y).X(x) \tag{5}$$

Substituting in Eq.3 and separating the variables, we get

$$\frac{d^2Y(y)}{dy^2} + \varepsilon'^2\lambda^2 Y = 0 \tag{6}$$

with $$\frac{dY(0)}{dy} = 0, \text{ and } \frac{dY(1)}{dy} = -H'\ Y(1)$$

and $$\frac{d^2X(x)}{dx^2} - \lambda^2 X = 0 \tag{7}$$

with $$\frac{dX(0)}{dx} = -1/Y, \text{ and } \frac{dX(1)}{dx} = -H_1\ X(1)$$

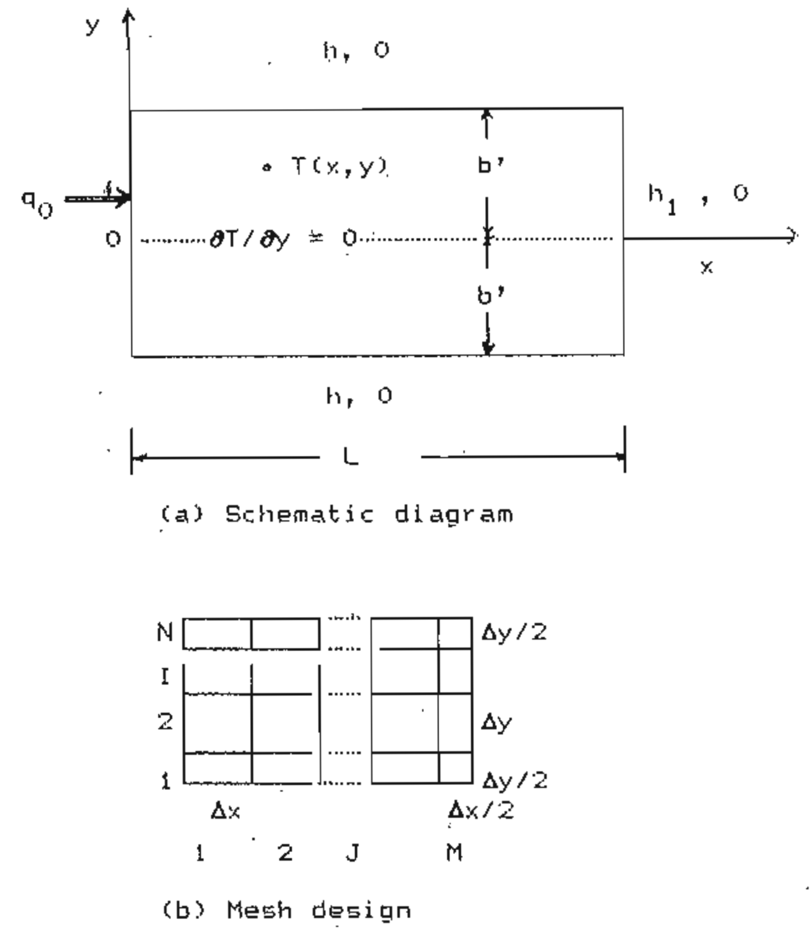

Fig. 1 Two-dimensional rectangular fin

Solution of Eq.6 is given by:

$$Y(y) = C_n \ \phi_n(y) \quad (8)$$

where $\phi_n(y) = \cos(\varepsilon' \lambda_n y)$

and the eigenvalues of λ_n are positive roots of the transcendental equation $\zeta_n \tan \zeta_n = hb'/k = Bi \ \varepsilon'$

where $\zeta_n = \varepsilon' \ \lambda_n$

Solution of Eq.7 is then given by:

$$X(x) = A \ e^{\lambda_n x} + B \ e^{-\lambda_n x} \quad (9)$$

Applying the homogeneous boundary condition at x = 1, Eq.9 becomes

$$X_n(x) = A \ (e^{\lambda_n x} + E \ e^{2\lambda_n} \ e^{-\lambda_n x}) \quad (10)$$

where $E = \dfrac{\lambda_n + (h_1/k)}{\lambda_n - (h_1/k)}$

Substituting Eqs.8 and 10 into Eq.5, we get

$$T(x,y) = \sum_{n=1}^{\infty} a_n \ e^{\zeta_n x/\varepsilon'} \left[1 + E \ e^{2\zeta_n (1-x)/\varepsilon'} \right] \cos \zeta_n y \quad (11)$$

where $a_n = C_n$ A. Applying the nonhomogeneous boundary condition in the x direction $\dfrac{\partial T(0,y)}{\partial x} = -1$, we obtain

$$-1 = \sum_{n=1}^{\infty} a_n \ (\zeta_n/\varepsilon')(1 - E \ e^{2\zeta_n/\varepsilon'}) \cos\zeta_n y \quad (12)$$

Using the property of orthogonality [1], the coefficients a_n may be calculated by the expression:

$$a_n \ (\zeta_n/\varepsilon')(1 - E \ e^{2\zeta_n/\varepsilon'}) = \frac{\int_0^1 (-1) \cos(\zeta_n y) \ dy}{\int_0^1 \cos^2(\zeta_n y) \ dy}$$

$$a_n = \left[\frac{-2 \sin\zeta_n}{\zeta_n + \sin\zeta_n \cos\zeta_n}\right] \left[\frac{1}{(1 - E \ e^{2\zeta_n/\varepsilon'})}\right] (\varepsilon'/\zeta_n) \quad (13)$$

where E is then given by: $E = \dfrac{\zeta_n + H_1 \varepsilon'}{\zeta_n - H_1 \varepsilon'}$

To obtain the transient temperature distribution for the specified problem, the finite difference technique is then applied. In the finite difference technique, the region of interest is descretized to M and N nodal points in the x and y directions as shown in Fig.1b. Energy balance is then applied to each nodal point and the dimensionless temperature of the nodal point i,j is obtained in the following form :

$$T_{i,j}^{t+\Delta t} = a_o + a_1 \ T_{i,j-1}^{t} + a_2 T_{i,j+1}^{t} + a_3 T_{i-1,j}^{t} + a_4 T_{i+1,j}^{t} + a_5 T_{i,j}^{t} \quad (14)$$

where the time interval $\Delta t = \Delta Fo = \frac{k\ \Delta t^*}{\rho c L^2}$.

The coefficients a_o, a_1, a_2, a_3, a_4, and a_5 which satisfy the boundary conditions (Eqs.4) are listed in the table below, where

$A = \Delta t\ M^2$, $A1 = \Delta t\ H_1\ M$, $B = \Delta t(N/\varepsilon)^2$, $C = 2(N/\varepsilon)\Delta t\ Bi$, and $D = M\ \Delta t$

Node	a_o	a_1	a_2	a_3	a_4	a_5
Interior	0	A	A	B	B	
Row I=1	0	A	A	C	2B	
Row I=M	0	A	A	2B	C	
Column J=1	D	0	A	B	B	
Column J=N	0	2A	2A1	B	B	$1 - a_1 - a_2 - a_3 - a_4$
Corner 1,1	D	0	A	C	2B	
Corner 1,N	0	2A	2A1	C	2B	
Corner M,1	D	0	A	2B	C	
Corner M,N	0	2A	2A1	2B	C	

The obtained set of algebraic equations are then solved using the simple explicit scheme [2]. Applying the boundary conditions of Eq.4b, one finds that the solution is stable if the following condition is satisfied :

$$\Delta t = \Delta Fo' \le \frac{1}{2(N^2 + Bi(M/\varepsilon) + (M/\varepsilon)^2)}$$

RESULTS AND DISCUSSION

Figure 2 illustrates the time development of temperature profiles as obtained from both the analytical steady-state solution (Eq.11) and the numerical solution (Eq.14) for adiabatic tip surface ($h_1 = 0$) and thickness/length ratio $\varepsilon = 2b'/L = 0.1$. The well agreement between the steady-state analytical temperature profile and the numerical obtained profile at $t = \infty$ confirms the validity of both solutions. The effect of Biot number on the temperature profile is shown on Fig.3 for the same tip condition and at time t = 2. The curves indicates that as the Biot number increases the temperature profile becomes lower. This is physically accepted because the high Bi means efficient convective cooling process of the fin. This result is also confirmed by examining the time development of temperature profiles for a high Biot number (Bi = 3) which are plotted on Fig.4. It is clear that the steady-state base temperature level is higher about 5.7 times for Bi =0.1 than that for Bi = 3.

The effect of thickness/length ratio ϵ on the transient temperature development is illustrated on Fig.5. Examining this figure, one can deduce that the level of temperature profiles for $\epsilon = 0.5$ is higher than the corresponding level for $\epsilon = 0.1$ under the same conditions. In addition, it is clear that the steady-state temperature profile is reached in longer time for $\epsilon = 0.5$ than for $\epsilon = 0.1$.
Figure 6 illustrates the effect of tip condition on the transient temperature, where the temperature profiles are plotted of $\epsilon = 0.5$ and $h_1 = h$. Comparing the curves on Fig.5 and Fig.6 which are plotted for different tip condition ($h_1 = 0$, and $h_1 = h$), one can conclude that the efficient convective cooling of the tip surface reduces the level of temperature profiles, which is physically expected.
The steady-state temperature profiles for different Biot numbers are given on Fig.7. Results show that there is a little difference between the surface and center temperature distributions of a fin for small Bi and small ϵ. This difference increases as Bi and ϵ become higher as can be deduced from the curves on Figs.7 and 8. Figure 9 describes the temperature profiles for a fin which represent an absorber plate in a flat plate solar collector ($\epsilon = 10$). Finally, the dimensionless Eq.14 indicates that the proposed finite difference technique is simple and very fast even for large time t.

CONCLUSIONS

From the above discussion it can be concluded that the obtained analytical steady-state temperature profile and the proposed simple dimensionless finite difference technique correctly predicts the time development of temperature profiles for transient conduction in a 2-D rectangular fin which is subjected to a constant heat flux at the base with different fin tip conditions.

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NOMENCLATURE

$2b'$ fin thickness

c specific heat

h heat transfer coefficient

h_1 tip heat transfer coefficient

k thermal conductivity

L fin length

t^* real time

T^* temperature

T^*_∞ ambient temperature

T^*_o temperature at the fin base

x^* longitudinal coordinate

y^* transverse direction

Dimensionless groups

Bi Biot number = hL/k

Fo Fourier number = $k\,t^*/(\rho c L^2)$

$x = x^*/L$

$y = y/b'$

$t = k\,t^*/(\rho c L^2)$

$T = (T^*-T^*_\infty)/(q_o L/k)$

$H' = hb'/k$

$H_1 = h_1 L/k$

Greek symbols

$\epsilon = 2b'/L$

$\epsilon' = b'/L$

ρ density

λ eigenvalue

ζ eigenvalue

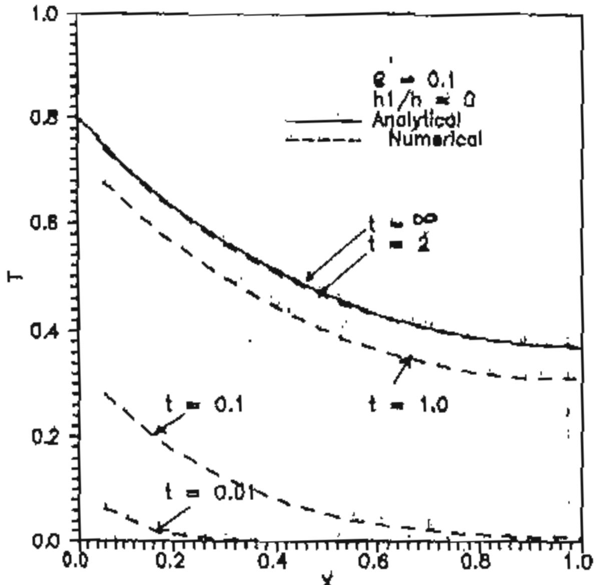

Fig.(2): Time development of temperature profiles for Bi = 0.1

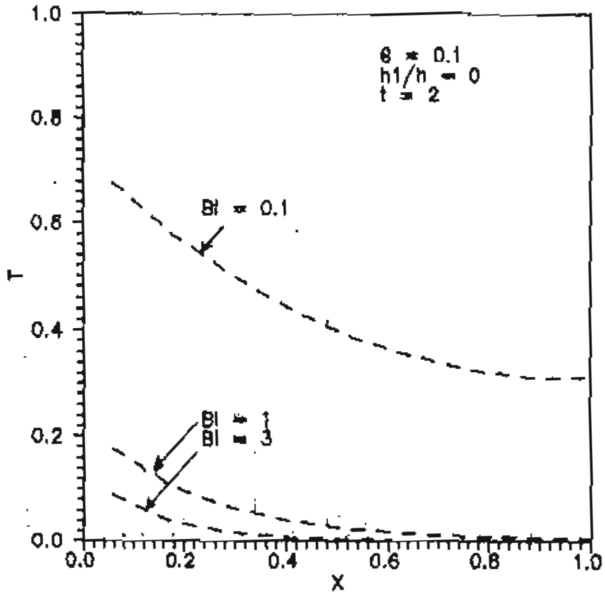

Fig.(3) Effect of Biot number on the temperature profile at t = 2

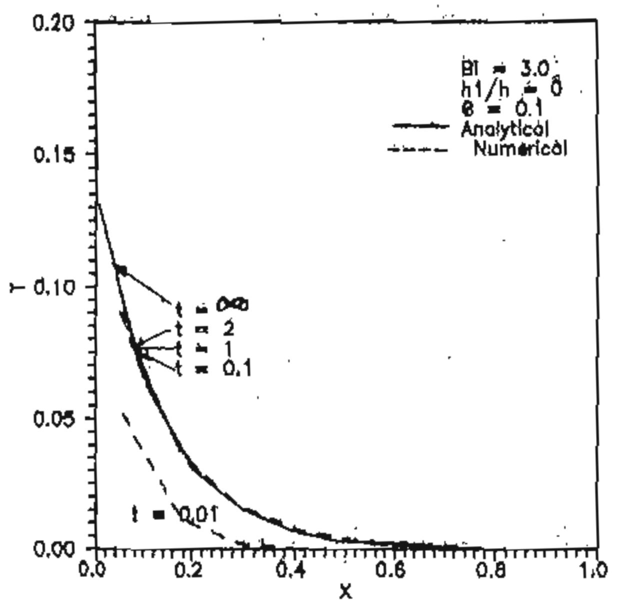

Fig.(4) Time development of temperature profiles for Bi = 3.0

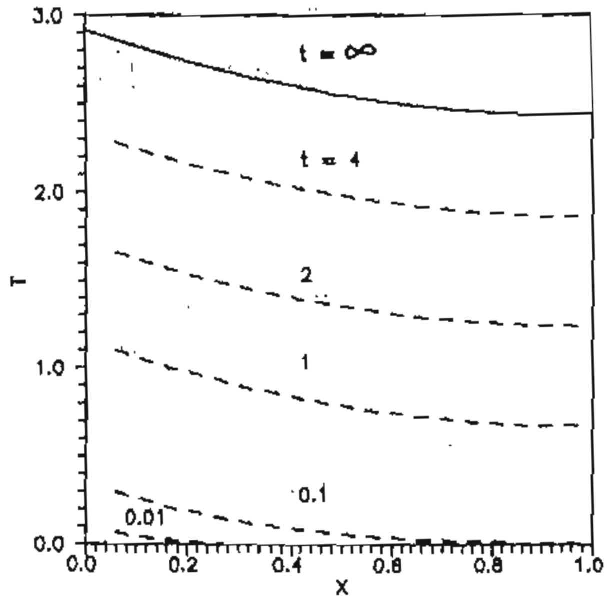

Fig.(5) Time development of temperature profiles for Bi=0.1, θ = 0.5, h1/h = 0

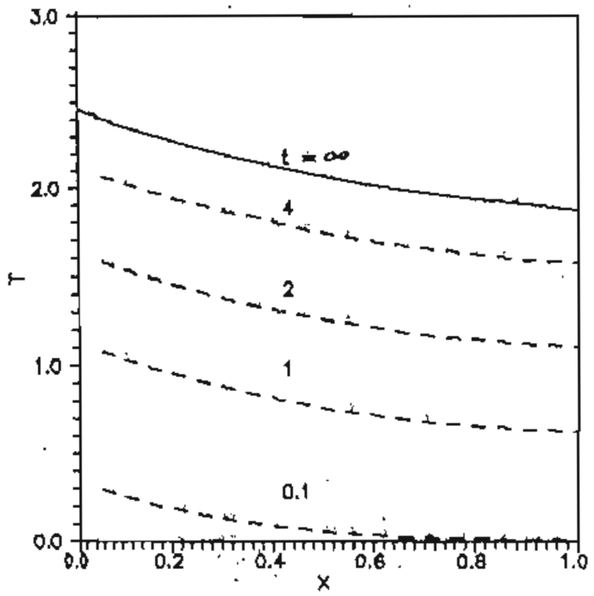

Fig.(6) Time development of temperature profiles for Bi = 0.1, β = 0.5, h1/h = 1

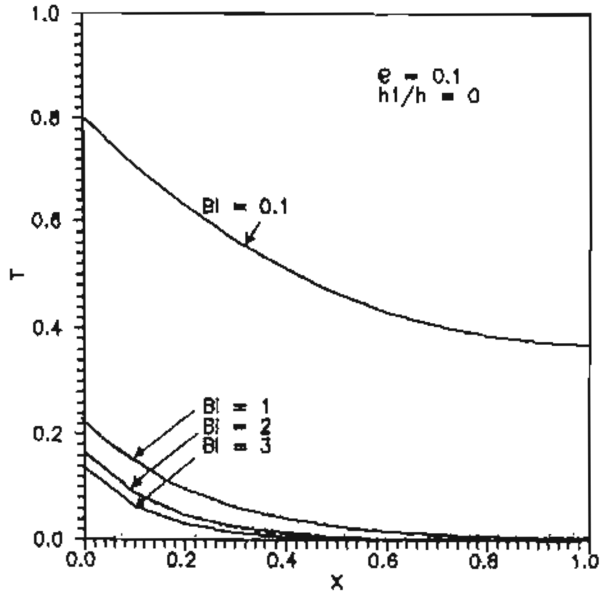

Fig.(7) Steady-state temperature profiles for different Biot numbers

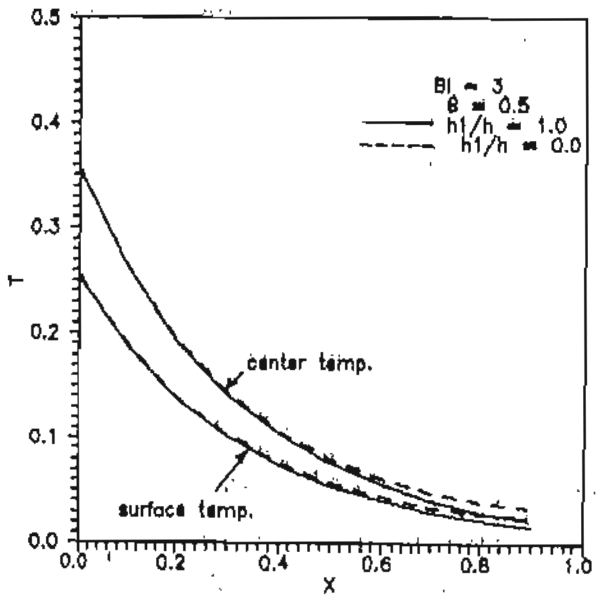

Fig.(8) Center and surface temperature profiles for different tip conditions

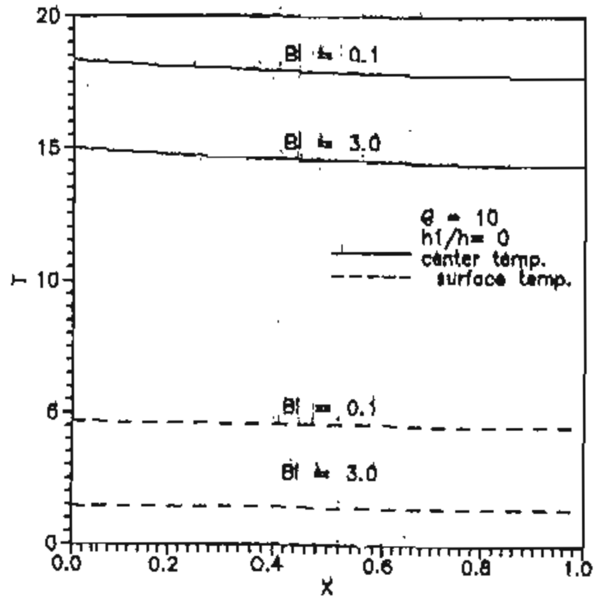

Fig.(9) Steady-state temperature profiles for β = 10